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# An Estimate for Lipschitz Constants of Metric Projections*

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The main purpose of the paper is to give a global estimate for Lipschitz constants of metric projections in a $p$-uniformly convex and $q$-uniformly smooth Banach space. © 1999 Academic Press

## 1. INTRODUCTION

Metric projection operators $P_\Omega$ on convex closed sets $\Omega$ (in the sense of best approximations) are widely used in theoretical and applied areas of mathematics, especially connected with problems of optimization and approximation. As examples one can consider iterative-projective methods for solving equations, variational inequalities and minimizations of functionals [1], and methods of alternating projections for finding common points of convex closed sets in Hilbert space [4, 5, 7].

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It is well known [9] that in a Hilbert space $X$ the following estimate holds:

$$\|P_\Omega(x) - P_\Omega(y)\| \le \|x - y\|, \quad \text{for all } x, y \in X, \tag{1}$$

and many problems can be solved by applying the inequality (1). Therefore, one natural method to solve problems in Banach spaces $X$ is to establish estimates analogous to (1). This is in fact shown by some author's recent work; see, for example, Xu and Roach [11]. Alber and Notik [3], and Alber [2].

However, applying their results to a Hilbert space $X$, we can only obtain the following estimate:

$$\|P_\Omega(x) - P_\Omega(y)\| \le c\|x - y\|, \quad \text{for all } x, y \in X, \tag{2}$$

for some constant $c > 1$. Comparing (2) with (1), it is natural to ask whether there exists a common estimate for $\|P_\Omega(x) - P_\Omega(y)\|$ in a uniformly convex and uniformly smooth Banach space such that (1) holds for a Hilbert space. The purpose of the present paper is to give such an estimate for $\|P_\Omega(x) - P_\Omega(y)\|$ in a $p$-uniformly convex and $q$-uniformly smooth Banach space.

## 2. PRELIMINARIES

Now let us recall the definition of the metric projection operator. Let $\Omega$ be a convex closed subset in a Banach space $X$. For any $x \in X$ an element $\bar{x} \in \Omega$ such that $\|x - \bar{x}\| = \min_{m \in \Omega}\|x - m\|$ is called a nearest point to $x$. If for each $x \in X$, it has a unique nearest point in $\Omega$, then define $P_\Omega(x)$ to be the nearest point of $x$. It is known that $P_\Omega$ is well defined in a uniformly convex Banach space. The moduli of convexity and smoothness of $X$ are defined, respectively, by

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\tfrac{1}{2}(x + y)\right\|: \|x\| = \|y\|, \|x - y\| = \epsilon\right\}, \quad 0 \le \epsilon \le 2,$$

and

$$\rho_X(t) = \sup\left\{\tfrac{1}{2}(\|x + y\| + \|x - y\|) - 1: \|x\| = 1, \|y\| = t\right\}, \quad t > 0.$$

Let $p \ge 2$, $1 < q \le 2$, $X$ is said to be $p$-uniformly convex (resp. $q$-uniformly smooth) if there is a constant $d > 0$ such that $\delta_X(\epsilon) \ge d\epsilon^p$ (resp. $\rho_X(t) \le dt^q$).

Let

$$d_p = \inf\left\{\frac{(1/2)\|x\|^p + (1/2)\|y\|^p - \|(x+y)/2\|^p}{\|(x-y)/2\|^p} : \|x-y\| > 0\right\},$$

$$c_q = \sup\left\{\frac{(1/2)\|x\|^q + (1/2)\|y\|^q - \|(x+y)/2\|^q}{\|(x-y)/2\|^q} : \|x-y\| > 0\right\}.$$

From [10] we have

PROPOSITION 1. *Let $p > 1$. Then a Banach space $S$ is $p$-uniformly convex if and only if $d_p > 0$.*

PROPOSITION 2. *Let $q > 1$. Then a Banach space $X$ is $q$-uniformly smooth if and only if $c_q > 0$.*

The following proposition is simple and well known.

PROPOSITION 3. *Let $q > 1$ and $X$ be strictly convex. Then for any $x, y \in X$, $f_x \in J_q(x)$, $f_y \in J_q(y)$, there holds*

$$\langle x - y, f_x - f_y \rangle > 0$$

*where* $J_q(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}$.

In order to give the main theorem we need a lemma about the constants $d_p$ and $c_q$.

LEMMA 1. *Let $X$ be a Banach space and $n$ be a positive integer. Then for any $x, y \in X$ the following hold*

$$\left\|\left(1 - \frac{1}{2^n}\right)x + \frac{1}{2^n}y\right\|P \le \left(1 - \frac{1}{2^n}\right)\|x\|^p + \frac{1}{2^n}\|y\|^p - d_p\left\{\sum_{i=0}^{n-1}\left(\frac{1}{2^p}\right)^i\left(\frac{1}{2}\right)^{n-1-i}\right\}\left\|\frac{x-y}{2}\right\|^p \tag{3}$$

$$\left\|\left(1 - \frac{1}{2^n}\right)x + \frac{1}{2^n}y\right\|^q \ge \left(1 - \frac{1}{2^n}\right)\|x\|^q + \frac{1}{2^n}\|y\|^q - c_q\left\{\sum_{i=0}^{n-1}\left(\frac{1}{2^q}\right)^i\left(\frac{1}{2}\right)^{n-1-i}\right\}\left\|\frac{x-y}{2}\right\|^q. \tag{4}$$

*Proof.* We will prove formula (3) by induction while formula (4) can be proved similarly. From the definition of $d_p$, we have that for any $x, y \in X$,

$$\left\| \frac{1}{2}x + \frac{1}{2}y \right\|^p \le \frac{1}{2}\|x\|^p + \frac{1}{2}\|y\|^p - d_p \left\| \frac{x-y}{2} \right\|^p.$$

This means that (3) holds for $n = 1$. Now assume that (3) holds for $n = k$. Let us prove that (3) holds for $n = k + 1$. In fact, for any $x, y \in X$,

$$\begin{aligned}
&\left\| \left(1 - \frac{1}{2^{k+1}}\right)x + \frac{1}{2^{k+1}}y \right\|^p \\
&\quad = \left\| \frac{1}{2}x + \frac{1}{2}\left[\left(1 - \frac{1}{2^k}\right)x + \frac{1}{2^k}y\right] \right\| \\
&\quad \le \frac{1}{2}\|x\|^p + \frac{1}{2}\left\| \left(1 - \frac{1}{2^k}\right)x + \frac{1}{2^k}y \right\|^p - d_p \left(\frac{1}{2^k}\right)^p \left\| \frac{x-y}{2} \right\|^p \\
&\quad \le \frac{1}{2}\|x\|^p + \frac{1}{2}\left[\left(1 - \frac{1}{2^k}\right)\|x\|^p + \frac{1}{2^k}\|y\|^p\right] \\
&\qquad - \frac{1}{2}d_p \left\{ \sum_{i=0}^{k-1} \left(\frac{1}{2^p}\right)^i \left(\frac{1}{2}\right)^{k-1-i} \right\} \left\| \frac{x-y}{2} \right\|^p - d_p \left(\frac{1}{2^p}\right)^k \left\| \frac{x-y}{2} \right\|^p \\
&\quad = \left(1 - \frac{1}{2^{k+1}}\right)\|x\|^p + \frac{1}{2^{k+1}}\|y\|^p - d_p \left\{ \sum_{i=0}^{k} \left(\frac{1}{2^p}\right)^i \left(\frac{1}{2}\right)^{k-i} \right\} \left\| \frac{x-y}{2} \right\|^p.
\end{aligned}$$

Hence (3) holds for $n = k + 1$ and the proof is complete. ∎

PROPOSITION 4. *Let $q > 1$ and $X$ be $q$-uniformly smooth. Then for any $x, y \in X$ the following inequality holds*

$$\|x + y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + \frac{c_q}{2^{q-1} - 1}\|y\|^q. \tag{5}$$

*Proof.* From (4), it follows that for any $n > 1$

$$\begin{aligned}
&\frac{\|x + (1/2^n)y\|^q - \|x\|^q}{1/2^n} \\
&\qquad = \frac{\|(1 - (1/2^n))x + (1/2^n)(x + y)\|^q - \|x\|^p}{1/2^n} \\
&\qquad \ge \|x + y\|^q - \|x\|^q - \frac{c_q(1 - 2^{n-nq})}{2^{q-1} - 1}\|y\|^q.
\end{aligned}$$

Thus we have that

$$q\langle y, J_q(x)\rangle = \lim_{n\to\infty} \frac{\|x + (1/2^n)y\|^q - \|x\|^q}{1/2^n}$$
$$\geq \|x+y\|^q - \|x\|^q - \frac{c_q}{2^{q-1}-1}\|y\|^q.$$

This proves the proposition. ∎

## 3. MAIN THEOREM

Now we are ready to prove the main theorem

THEOREM 1. *Let $X$ be a $p$-uniformly convex and $q$-uniformly smooth Banach space. Let $\Omega$ be a closed convex subset in $X$. Then for any $x, y \in X$ there holds*

$$\|P_\Omega x - P_\Omega y\| \leq \left[\frac{pc_q(2^{p-1}-1)}{qd_p(2^{q-1}-1)}c^{p-q}\right]\|x-y\|^{q/p}$$

*where* $c = \max\{\|x - P_\Omega y\|, \|y - P_\Omega x\|\}$.

*Proof.* Let

$$Px = P_\Omega x, \qquad Py = P_\Omega y,$$

and

$$x_n = Px + 2^n(x - Px)$$

Then $Px_n = Px$ for any positive integer $n$. It follows that

$$\|x_n - Px\|^p \leq \|x_n - Py\|^p \leq \|x - Py + (2^n - 1)(x - Px)\|^p.$$

Dividing the inequality by $2^n$ and using Lemma 1, we have

$$\frac{1}{2^n}\|x - Px\|^p \leq \frac{1}{2^n}\|x - Py\|^p - d_p\left\{\sum_{i=0}^{n-1}\left(\frac{1}{2^p}\right)^i\left(\frac{1}{2}\right)^{n-1-i}\right\}\left\|\frac{Px - Py}{2}\right\|^p.$$

Multiplying the inequality by $2^n$ and letting $n \to \infty$, we obtain

$$\frac{d_p}{2^{p-1}-1}\|Px - Py\|^p \leq \|x - Py\|^p - \|x - Px\|^p.$$

Using Cauchy mean-valued theorem we obtain

$$\frac{d_p}{2^{p-1}-1}\|Px-Py\|^p \le \frac{p}{q}c^{p-q}\big(\|x-Py\|^q - \|x-Px\|^q\big).$$

Similarly, we also have

$$\frac{d_p}{2^{p-1}-1}\|Px-Py\|^p \le \frac{p}{q}c^{p-q}\big(\|y-Px\|^q - \|y-Py\|^q\big).$$

It follows that

$$\begin{aligned}2\frac{d_p}{2^{p-1}-1}\|Px-Py\|^p \le \frac{p}{q}c^{p-q}\big(&\|x-Py\|^q - \|x-Px\|^q \\ &+\|y-Px\|^q - \|y-Py\|^q\big).\end{aligned}$$

Note that (5) implies

$$\|x-Py\|^q - \|y-Py\|^q \le q\langle x-y, J_q(y-Py)\rangle + \frac{c_q}{2^{q-1}-1}\|x-y\|^q,$$

$$\|y-Px\|^q - \|x-Px\|^q \le q\langle y-x, J_q(x-Px)\rangle + \frac{c_q}{2^{q-1}-1}\|x-y\|^q.$$

Hence

$$\begin{aligned}&2\frac{d_p}{2^{p-1}-1}\|Px-Py\|^p \\ &\quad\le \frac{p}{q}c^{p-q}\big(q\langle x-y, J_q(y-Py)\rangle + q\langle y-x, J_q(x-Px)\rangle\big) \\ &\qquad + 2\frac{c_q}{2^{q-1}-1}\|x-y\|^q.\end{aligned}$$

From the characterization of a best approximation it follows that

$$\langle Px-Py, J_q(y-Py)\rangle \le 0, \qquad \langle Py-Px, J_q(x-Px)\rangle \le 0.$$

This, together with Proposition 3, gives

$$\begin{aligned}&2\frac{d_p}{2^{p-1}-1}\|Px-Py\|^p \\ &\quad\le 2\frac{p}{q}c^{p-q}\left(q\langle Qx-Qy, J_q(Qy)-J_q(Qx)\rangle + \frac{c_q}{2^{q-1}-1}\|x-y\|^q\right) \\ &\quad\le 2\frac{p}{q}c^{p-q}\,\frac{c_q}{2^{q-1}-1}\|x-y\|^q\end{aligned}$$

where $Qx = x - Px$. That is,

$$\|Px - Py\| \le \left[ \frac{pc_q(2^{p-1} - 1)}{qd_p(2^{q-1} - 1)} c^{p-q} \right] \|x - y\|^{q/p}.$$

The proof is complete. ∎

If $X$ is a Hilbert space then $p = q = 2$ and $d_p = c_q = 1$. Therefore we have

COROLLARY 1. *Let $X$ be a Hilbert space and $\Omega$ be a convex closed subset in $X$. Then the following holds*

$$\left\| P_\Omega(x) - P_\Omega(y) \right\| \le \|x - y\|, \qquad \textit{for all } x, y \in X.$$

COROLLARY 2. *Let $X$ be the $L_r(\mu)$-space and $\Omega$ be a convex closed subset in $X$. Then the following holds*

$$\left\| P_\Omega(x) - P_\Omega(y) \right\| \le \begin{cases} \dfrac{r(r-1)c^{r-2}(2^{r-1} - 1)}{2} \|x - y\|^{2/r}, & r \ge 2 \\[2ex] \dfrac{2c^{2-r}}{r(r-1)(2^{r-1} - 1)} \|x - y\|^{r/2}, & r \le 2. \end{cases}$$

*Proof.* We divide two cases, $r \ge 2$ and $r \le 2$, to prove the corollary.

(i) If $r \ge 2$, then $p = r, q = 2$, that is, $L_r(\mu)$ is $r$-uniformly convex and 2-uniformly smooth. It follows from [6] that $d_p = d_r = 1, c_q = c_2 \le p - 1$. Thus the result follows from Theorem 1.

(ii) If $r \le 2$, then $p = 2, q = r$, that is, $L_r(\mu)$ is 2-uniformly convex and $r$-uniformly smooth. From [8] we have $d_p = d_2 \ge r - 1$. Now let us show that $c_q = c_r = 1$.

For $-1 \le t < 1$, let

$$f(t) = \frac{1/2 + (1/2)|t|^r - ((1+t)/2)^r}{((1-t)/2)^r}.$$

Then the derivative of $f(t)$ for $t \ne 0$ is

$$f'(t) = \frac{r\left(|t|^{r-1}s(t) - 2((1+t)/2)^{r-1} + 1\right)}{4((1-t)/2)^{r+1}}$$

where $s(t) = t/|t|$ for $t \neq 0$ and $s(0) = 0$. Write

$$g(t) = |t|^{r-1}s(t) - 2\left(\frac{1+t}{2}\right)^{r-1} + 1.$$

Then the derivative of $g(t)$ for $t \neq 0$ is

$$g'(t) = (r-1)\left(|t|^{r-2} - \left(\frac{1+t}{2}\right)^{r-2}\right).$$

Thus

$$\begin{aligned} g'(t) &> 0 \qquad \forall t \in (-1/3, 0) \cup (0, 1) \\ g'(t) &< 0 \qquad \forall t \in (-1, -1/3). \end{aligned}$$

From $g(1) = g(-1) = 0, g(0) < 0$ it follows that $g(t) \leq 0$ for all $t \in [-1, 1]$. This implies that $f'(t) \leq 0$ for all $t \in [-1, 0) \cup (0, 1)$ so that

$$\sup\{f(t): -1 \leq t < 1\} = f(-1) = 1.$$

Thus we have

$$\sup\left\{\frac{(1/2)|x|^r + (1/2)|y|^r - |(x+y)/2|^r}{|(x-y)/2|^r} : x \neq y\right\} = 1.$$

This implies that $c_q = c_r = 1$ and the proof is complete. ∎

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